PROOF OF A FORMULA DUE TO CAUCHY.

By Prof. W. H. Echols, Charlottesville, Va.

The function

$$\begin{vmatrix} 1 & , 1, & 1 & , 1 & , \ldots, & 1 \\ a^p x & , 1, & a & , a^2 & , \ldots, & a^{n-1} \\ (a^p x)^2 & , 1, & a^2 & , (a^2)^2 & , \ldots, & (a^{n-1})^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (a^p x)^n & , 1, & a^n & , (a^2)^n & , \ldots, & (a^{n-1})^n \end{vmatrix} \div \zeta^{\frac{1}{2}}(a, a^2 \ldots a^n), \tag{1}$$

vanishes when $x = a^{i-p}$ $(i = 0, 1, \ldots, n - 1)$, p being any finite quantity.

If we expand (1) with respect to the first column, we have

$$1 + \sum_{r=1}^{r=n} (-1)^r \frac{(1 - a^{-n+r-1}) \ldots (1 - a^{-n})}{(a - 1) \ldots (a^r - 1)} a^{r(p+1)} x^r .$$

This rational integral function of x of the nth degree has the n roots a^{-p}, $a^{1-p}, \ldots, a^{n-1-p}$; whence we have the identity

$$\prod_{i=0}^{i=n-1} (1 - a^{p-i} x) = 1 + \sum_{r=1}^{r=n} (-1)^r \frac{(1 - a^{-n+r-1}) \ldots (1 - a^{-n})}{(a - 1) \ldots (a^r - 1)} a^{r(p+1)} x^r , \tag{2}$$

Evidently, if $a > 1$, we have

$$\prod_{i=0}^{i=\infty} (1 - a^{p-i} x) = 1 + \sum_{r=1}^{r=\infty} \frac{(-1)^r a^{r(p+1)}}{(a - 1) \ldots (a^r - 1)} x^r . \tag{3}$$

Let $x = a^{q-p}$, where q is any positive integer or zero, then

$$0 = 1 + \sum_{r=1}^{r=\infty} \frac{(-1)^r a^{r(q+1)}}{(a - 1) \ldots (a^r - 1)} .$$

In (2) let $p = -1$, we then have Cauchy's formula (Comptes Rendus, 1840. Chrystal's Algebra, II, 316), and (3) becomes Euler's theorem (Introd. in Anal. Inf., § 306).

As a consequence the sum of the products, taken q at a time without repetition, of the quantities

$$a^{-i} \quad (i = 1, 2, 3, \ldots, \infty)$$

is

$$\frac{1}{(a - 1) \ldots (a^q - 1)} .$$